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Displacing small particles by unsteady temperature fields

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A model for particle propulsion by an instantaneous heat discharge is presented. The flow is driven by a gravity-independent transient fluid dilatation, engendered by an unsteady temperature field which corresponds to heat emission from a localized source located within the particle. We focus on the highly eccentric case, where the heat is released in proximity to the particle surface. Solution of the Stokes equations and subsequent evaluation of the resulting hydrodynamic thrust yields a nonlinear non-autonomous ordinary differential equation governing the evolution of particle position with time. This equation depends upon a single parameter which represents the relative effects of heating magnitude and initial geometry.

1. Introduction

The miniaturization revolution in science and technology is posing significant engineering challenges in various fields of mechanics and applied physics. Among them is the controlled animation of fluid motion at the sub-micron scale, and specifically the delicate propulsion and positioning of small bodies which are immersed in liquids. These bodies may be either biological cells or the envisioned man-made micro-robots (Iddan *et al.* 2000).

Conventional flow and propulsion mechanisms, which abound at the 'macroscopic' scales, are often ineffective at micron scales and below. In pressure-driven flows, for example, the characteristic velocity varies as the square of characteristic length L, resulting in rapidly diminishing performance as L shrinks. The desire for simple yet effective flow mechanisms at the microscale has led to the use of innovative engineering devices (see Stone, Stroock & Ajdari 2004) which employ a multitude of physicochemical phenomena, such as electrokinetics, capillarity, and wettability. An attractive feature of these phenomena is the scaling with length dimension: the magnitude of electrokinetic flows, for example, is roughly independent of L. Of course, physicochemical effects can only be used in specific systems which possess the necessary 'ingredients': free interface for surface-tension driven flows, electrolyte solution for electrokinetic flows, etc.

Here we propose the use of a different phenomenon, exploiting the thermal expansion of fluids. Specifically, we focus upon *time-dependent* density variations occurring in liquids whose temperature fields vary with time (say, by virtue of an unsteady heating process). Because of the requirement of mass conservation, such variations are automatically accompanied by fluid motion. In contrast to buoyancy-driven free-convection flows (also driven by thermal expansion) which are animated by the dynamic action of gravity, the unsteady mechanism described herein is purely kinematic (and is independent of the presence of gravity). Being robust and relatively insensitive to the properties (polarity, wettability) of the ambient liquid, the 'unsteady-density' mechanism may therefore prove attractive in various applications, even compared with its physicochemical counterparts. The physical features of flows driven by time-dependent temperature fields have been recently discussed by Yariv & Brenner (2004).

Here we consider the possibility of using unsteady expansion to propel a rigid particle in an otherwise stationary liquid medium. The requisite heating may evolve naturally from existing processes within the particle (e.g. Joule heating), or may be deliberately generated. The purpose of this paper is to present the qualitative aspects of this proposed propulsion mechanism using a simple model problem. Since heat transport is a scalar process with no preferred direction in space, the use of flows driven by unsteady thermal expansion for propulsion objectives requires an asymmetric geometry. We here address what is perhaps the simplest model possessing such asymmetry: the propelled body is taken to be a rigid sphere of uniform thermal properties, and the unsteady heating is modelled by an instantaneous release of a concentrated amount of heat at an internal point of the sphere (located off centre). The emitted heat generates a transient temperature disturbance relative to the pre-existing uniform value. While this disturbance (together the concomitant fluid and particle motion) eventually fades away, the asymmetry guarantees a residual particle displacement, the calculation of which constitutes the main goal of this paper.

2. Problem formulation

Consider a stationary sphere of radius a positioned in an unbounded stationary fluid domain. In this reference state, the system is at mechanical and thermodynamic equilibrium; the quiescent fluid possesses uniform pressure and density values, say p_∞ and ρ_∞, and the temperature in the entire space is given by the uniform value T_∞. At time $t=0$ a finite amount Q of heat is released at an interior point of the particle at a distance d from the sphere's surface ($0<d<a$), engendering a transient temperature disturbance. As a result of the fluid's thermal expansion, this disturbance is accompanied by a density perturbation; owing to the requirement of mass conservation, a transient flow field is established.

The transport and flow processes are conveniently described in an inertial coordinate system fixed with respect to the distant fluid. Its origin O is chosen as the heat release point, and its Z-axis is taken to be the line connecting O to the sphere's centre. Owing to the axial symmetry of this configuration, the fluid motion produces a hydrodynamic force on the particle which is aligned along the Z-axis. Accordingly, the particle's time-dependent translational velocity adopts the form $\boldsymbol{U}(t)=\hat{\boldsymbol{z}}\,U(t)$. After attenuation of the temperature disturbance the fluid's velocity diminishes, as does $U(t)$. Nevertheless, the entire transport process results in a net translation of the sphere along the Z-direction, namely $\int_0^\infty U(t)\,\mathrm{d}t$. This displacement is conceptually measured by following a particle-fixed reference point P located at the sphere surface (denoted here by $\mathscr{S}$). In what follows it is convenient to choose P as the sphere's 'negative' pole, whose time-dependent Z-coordinate, say $Z_-(t)$, satisfies $Z_-(0)=-d$. The accumulated particle's displacement is therefore given by $Z_-(\infty)-Z_-(0)$. A schematic of the particle and coordinate system, at times $t=0$ and $t>0$, is provided in figure 1.

Given the small values of the thermal expansion coefficient in liquids (about $10^{-4}\,\mathrm{K}^{-1}$ for water), the heat transport and the concomitant flow are governed by a linearized equation set appropriate to weak thermal forcing (Yariv & Brenner 2004).

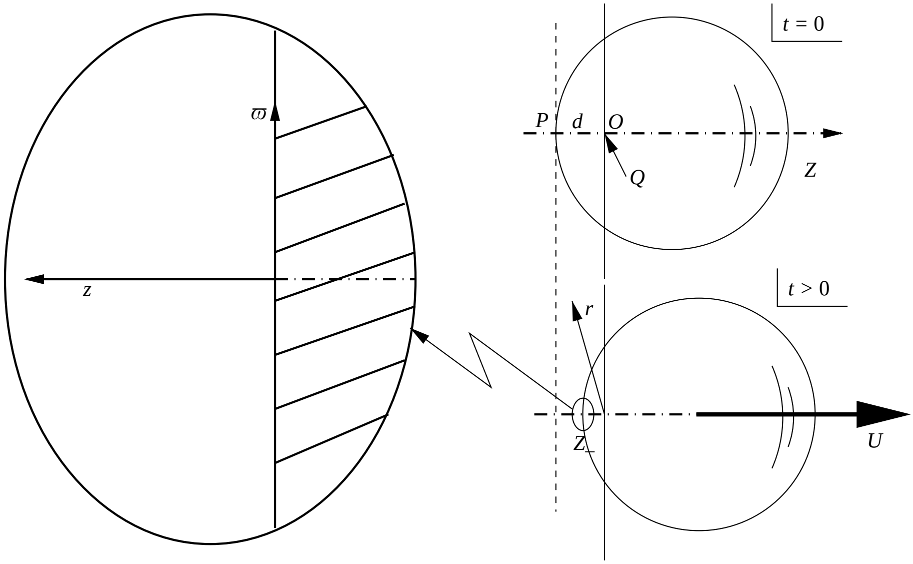

FIGURE 1. Schematic of the sphere geometry.

Thus, the temperature field obeys the linear conduction equation

$$\rho_\infty c_p \frac{\partial T}{\partial t} = k\nabla^2 T + q(\boldsymbol{x}, t), \tag{2.1}$$

in which c_p and k respectively denote the isobaric specific heat and thermal conductivity of the fluid in its reference state, and $q(\boldsymbol{x}, t)$ is a heat-source term. The resulting fluid expansion is governed by a Boussinesq-type linearization about the reference state, $\rho - \rho_\infty = -\beta\rho_\infty(T - T_\infty)$, with β denoting the thermal expansion coefficient evaluated at the reference state. This expansion animates fluid motion according to the linearized mass-conservation equation

$$\nabla \cdot \boldsymbol{v} = \beta \frac{\partial T}{\partial t}, \tag{2.2}$$

which is supplemented by the compressible Stokes equation,

$$\mu\left[\nabla^2 \boldsymbol{v} + \tfrac{1}{3}\nabla(\nabla \cdot \boldsymbol{v})\right] = \nabla p. \tag{2.3}$$

Here, μ is the dynamic viscosity of the liquid evaluated at the reference state. (Note that the quasi-steady variant of the momentum balance is used: for flows driven by diffuse thermal processes, this step is equivalent to the assumption of a large Prandtl number.) In principle, all fluid properties may depend upon the temperature (as does the density); however, consistency with the linearization scheme necessitates that those appearing in (2.1)–(2.3) are evaluated at the reference state.† The source term in (2.1) is modelled by the impulsive distribution

$$q(\boldsymbol{x}, t) = Q\, \delta(\boldsymbol{x})\delta(t). \tag{2.4}$$

Here, $\delta(\cdots)$ denotes Dirac's delta function.

† While the viscosity of liquids (as well as other transport properties) may have a stronger dependence on temperature than the density, this dependence would only appear in higher-order corrections: viscosity variations, *per se*, do not animate fluid motion, and can only modify the expansion-driven flow.

In the absence of external force fields, the hydrodynamic force acting on the particle must vanish:

$$\boldsymbol{F} = \oint_{\mathscr{S}} \mathrm{d}A\, \boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{0}. \tag{2.5}$$

Here, $\boldsymbol{n}$ denotes an outward-pointing unit vector normal to $\mathscr{S}$, and $\boldsymbol{\sigma}$ is the Newtonian stress tensor:

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \mu\left[\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^{\dagger} - \tfrac{2}{3}(\nabla\cdot\boldsymbol{v})\boldsymbol{I}\right]. \tag{2.6}$$

Since only the derivatives of p and T appear in (2.1)–(2.3), we hereafter use these symbols to denote the respective disturbances (relative to p_∞ and T_∞) generated by the heating process. (The hydrodynamic force (2.5) is invariant under the addition of a uniform value to the pressure field.) Thus, all flow variables – $\boldsymbol{v}$, p, and T – attenuate at large distances from the particle. On the surface $\mathscr{S}$ the fluid adheres to the particle, $\boldsymbol{v} = \boldsymbol{U}(t)$, whereas the appropriate boundary conditions governing the temperature field depend upon the specific thermal model of the particle's material.

The flow and transport problems depend upon the instantaneous particle position, which is embodied in the value of $Z_{-}(t)$. This coordinate is governed by the kinematic equation,

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = U(t), \tag{2.7}$$

which serves to 'close' the governing equations.

3. Heat transport and flow

Since our interest is in the simplest model possible, we consider the case where the particle possesses the same thermal conductivity and diffusivity as the fluid. Accordingly, the temperature field is not 'aware' of the presence of a particle, and (2.1) applies in the entire space. The solution to that equation, which attenuates far from the particle, is given by the free-space Green function of the diffusion equation:

$$T = \frac{Q}{8\rho_\infty c_p}\frac{1}{(\pi\alpha t)^{3/2}}\exp\left(\frac{-r^2}{4\alpha t}\right). \tag{3.1}$$

Here, $r = |\boldsymbol{x}|$ is the radial coordinate in a spherical coordinate system centred about O (see figure 1) and $\alpha = k/\rho_\infty c_p$ is the thermal diffusivity of the fluid in its reference state.

Owing to the linearity of the hydrodynamic problem, it is convenient to decompose it into three separate components. The first, $(\boldsymbol{v}^{\mathrm{I}}, p^{\mathrm{I}})$, is driven by the thermal expansion process,

$$\nabla\cdot\boldsymbol{v}^{\mathrm{I}} = \beta\frac{\partial T}{\partial t}, \tag{3.2}$$

but does not take account of the presence of the particle. The inevitable violation of the boundary conditions on $\mathscr{S}$ is rectified by the 'reflected' field, $(\boldsymbol{v}^{\mathrm{II}}, p^{\mathrm{II}})$. This flow field is solenoidal, and satisfies $\boldsymbol{v}^{\mathrm{II}} = -\boldsymbol{v}^{\mathrm{I}}$ on $\mathscr{S}$. (Mathematically speaking, field I is a particular solution of (2.2), whereas field II is a homogeneous solution.) The combination of flows I and II describes thermally induced flow about a *stationary* particle. The third flow component, $(\boldsymbol{v}^{\mathrm{III}}, p^{\mathrm{III}})$, is also solenoidal, and satisfies $\boldsymbol{v}^{\mathrm{III}} = \boldsymbol{U}$ on $\mathscr{S}$. This field represents a pure translation of the particle with velocity $\boldsymbol{U}$. All three fields satisfy the momentum balance (2.3) and vanish at large distances away from the particle.

3.1. *Flow field I*

At $t = 0$ the origin is excluded from the fluid domain. Accordingly, the temperature in the fluid domain satisfies the homogeneous counterpart of (2.1):

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T. \tag{3.3}$$

Substitution of this equation into (3.2) readily yields

$$\nabla \cdot \boldsymbol{v}^{\mathrm{I}} = \beta \alpha \nabla^2 T. \tag{3.4}$$

A solution of this equation, which satisfies the attenuation requirement at infinity, is the potential flow field,

$$\boldsymbol{v}^{\mathrm{I}} = \beta \alpha \nabla T, \tag{3.5}$$

to which, through (2.3), the following pressure field corresponds:

$$p^{\mathrm{I}} = \tfrac{4}{3}\mu \nabla(\nabla \cdot \boldsymbol{v}^{\mathrm{I}}). \tag{3.6}$$

Substitution of (3.6) into (2.6) provides the following expression for the stress communicated by flow field I (since $\boldsymbol{v}^{\mathrm{I}}$ is irrotational, the tensor $\nabla \boldsymbol{v}^{\mathrm{I}}$ is symmetric):

$$\boldsymbol{\sigma}^{\mathrm{I}} = 2\mu[\nabla \boldsymbol{v}^{\mathrm{I}} - (\nabla \cdot \boldsymbol{v}^{\mathrm{I}})\boldsymbol{I}]. \tag{3.7}$$

Finally, substitution of (3.1) into (3.5) yields the explicit expression for the spherically symmetric field $\boldsymbol{v}^{\mathrm{I}} = \hat{\boldsymbol{r}}\, v_r(r)$ (where $\hat{\boldsymbol{r}}$ is the unit vector in the radial direction):

$$v_r = -\frac{l^3 r}{16(\pi \alpha t)^{3/2} t} \exp\left(\frac{-r^2}{4\alpha t}\right). \tag{3.8}$$

Here, l is an effective penetration length of the initial heat impulse, $l^3 = \beta Q/\rho_\infty c_p$.

We here focus upon the case of weak thermal forcing, whence $l \ll a$. Since the temperature field (and, consequently, the induced flow) decay exponentially fast away from O, it is expected that the excess stresses that contribute to the hydrodynamic thrust on the particle are significant only at a small part of $\mathscr{S}$, say $\overline{\mathscr{S}}$, centred about the negative pole P. It is therefore reasonable to consider the highly eccentric case where the heat is released in proximity to the surface, $d \ll a$. To leading-order terms in d/a, $\overline{\mathscr{S}}$ is approximated by the plane $Z = Z_-$. It is therefore convenient to employ a cylindrical coordinate system, (z, ϖ, ϕ), with $z = Z_- - Z$ and ϖ respectively denoting axial and radial coordinates (see figure 1) and ϕ denoting a (degenerate) azimuthal coordinate. In that system, the axial and radial velocity components of $\boldsymbol{v}^{\mathrm{I}}$ are respectively given by

$$w^{\mathrm{I}} = -v_r \frac{Z}{r} = -\frac{l^3 (z - Z_-)}{16(\pi \alpha t)^{3/2} t} \exp\left(\frac{-r^2}{4\alpha t}\right), \tag{3.9a}$$

$$u^{\mathrm{I}} = v_r \frac{\varpi}{r} = -\frac{l^3 \varpi}{16(\pi \alpha t)^{3/2} t} \exp\left(\frac{-r^2}{4\alpha t}\right), \tag{3.9b}$$

where $r = [(Z_- - z)^2 + \varpi^2]^{1/2}$.

The contribution $\boldsymbol{F}^{\mathrm{I}}$ of $(\boldsymbol{v}^{\mathrm{I}}, p^{\mathrm{I}})$ to the hydrodynamic force acting on the particle originates from stresses acting on $\overline{\mathscr{S}}$ (where, to leading-order terms, $\boldsymbol{n} = \hat{\boldsymbol{z}}$) and is therefore given by (cf. (3.7))

$$\boldsymbol{F}^{\mathrm{I}} = 2\mu \int_{z=0} \mathrm{d}A\, \hat{\boldsymbol{z}} \cdot [\nabla \boldsymbol{v}^{\mathrm{I}} - (\nabla \cdot \boldsymbol{v}^{\mathrm{I}})\boldsymbol{I}]. \tag{3.10}$$

It is obvious from symmetry considerations that this force is oriented along the z-axis, $\boldsymbol{F}^{\mathrm{I}} = \hat{\boldsymbol{z}}\, F^{\mathrm{I}}$. Accordingly, only the component $\hat{\boldsymbol{z}}\,\hat{\boldsymbol{z}}\,\partial w^{\mathrm{I}}/\partial z$ of $\nabla \boldsymbol{v}^{\mathrm{I}}$ contributes to $\boldsymbol{F}^{\mathrm{I}}$. Substitution of (3.1), (3.2), and (3.9) readily yields

$$F^{\mathrm{I}} = \frac{\mu l^3}{16(\pi\alpha t)^{3/2} t} \exp\left(\frac{-Z_-^2}{4\alpha t}\right) \int_{z=0} \mathrm{d}A \left(4 - \frac{\varpi^2}{\alpha t}\right) \exp\left(\frac{-\varpi^2}{4\alpha t}\right). \tag{3.11}$$

3.2. *Flow field II*

Flow field I results in normal and tangential velocity components on $\overline{\mathscr{S}}$, both violating the null boundary condition existing on the surface of a stationary particle. Consider now the 'reflected' flow field, $(\boldsymbol{v}^{\mathrm{II}}, p^{\mathrm{II}})$, which satisfies the incompressible Stokes equation,

$$\nabla \cdot \boldsymbol{v}^{\mathrm{II}} = 0, \tag{3.12a}$$

$$\mu \nabla^2 \boldsymbol{v}^{\mathrm{II}} = \nabla p^{\mathrm{II}}, \tag{3.12b}$$

together with attenuation conditions at infinity (when either z or ϖ becomes large). This field is driven by the boundary conditions

$$\boldsymbol{v}^{\mathrm{II}} = -\boldsymbol{v}^{\mathrm{I}} \quad \text{on} \quad \overline{\mathscr{S}}, \tag{3.13}$$

imposed so as to restore the impermeability and no-slip condition on $\overline{\mathscr{S}}$.

The resultant excess stresses are given by (cf. (2.6))

$$\boldsymbol{\sigma}^{\mathrm{II}} = -p\boldsymbol{I} + \mu[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^{\dagger}]. \tag{3.14}$$

As with field I, symmetry arguments necessitate that the force resulting from these stresses must be directed along the z-axis, $\boldsymbol{F}^{\mathrm{II}} = \hat{\boldsymbol{z}}\, F^{\mathrm{II}}$. Again, it is only the component $\hat{\boldsymbol{z}}\,\hat{\boldsymbol{z}}\,\partial w^{\mathrm{II}}/\partial z$ of $\nabla \boldsymbol{v}^{\mathrm{II}}$ that contributes to the force, leading to the expression

$$F^{\mathrm{II}} = \int_{z=0} \mathrm{d}A \left(-p^{\mathrm{II}} + 2\mu \frac{\partial w^{\mathrm{II}}}{\partial z}\right). \tag{3.15}$$

Since the 'inducer' field I is spherically symmetric (and, specifically, axisymmetric), it is clear that flow field II must be axisymmetric. Thus, the velocity vector $\boldsymbol{v}^{\mathrm{II}}$ comprises only axial and radial components, w^{II} and u^{II}, which are both independent of the azimuthal angle ϕ. In terms of the cylindrical coordinates, the continuity equation (3.12a) adopts the form

$$\frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi u^{\mathrm{II}}) + \frac{\partial w^{\mathrm{II}}}{\partial z} = 0, \tag{3.16}$$

while the respective axial and radial components of the momentum balance (3.12b) are

$$\mu \nabla^2 w^{\mathrm{II}} = \frac{\partial p^{\mathrm{II}}}{\partial z}, \tag{3.17}$$

$$\mu\left(\nabla^2 - \frac{1}{\varpi^2}\right) u^{\mathrm{II}} = \frac{\partial p^{\mathrm{II}}}{\partial \varpi}. \tag{3.18}$$

These equations, which are valid for $z>0$, are supplemented by the boundary condition (3.13), which adopts the form

$$\left.\begin{aligned} u^{\mathrm{II}} &= -u^{\mathrm{I}} \\ w^{\mathrm{II}} &= -w^{\mathrm{I}} \end{aligned}\right\} \qquad \text{at} \qquad z=0. \tag{3.19}$$

The preceding Stokes problem is solved by application of integral transforms. Specifically, we make use of the Hankel transform of order n (Sneddon 1972)

$$\widetilde{f}_n(\xi) \triangleq \mathscr{H}_n[f(\varpi);\varpi \longmapsto \xi] = \int_0^\infty \mathrm{d}\varpi\, \varpi f(\varpi) J_n(\varpi\xi), \tag{3.20}$$

with J_n being Bessel functions of the first kind. (With this definition, the relation between a function and its Hankel transform is symmetric.) The subsequent derivation is reminiscent of that appearing in a comparable geophysical model of viscous flow near a free surface (Haskell 1935; Sneddon 1972).

Forming the zeroth-order transform of (3.16)–(3.17) and the first-order transform of (3.18) yields the following system of ordinary differential equations:

$$\begin{aligned} \xi\widetilde{u}_1^{\mathrm{II}} + \frac{\mathrm{d}\widetilde{w}_0^{\mathrm{II}}}{\mathrm{d}z} &= 0, \\ \mu\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \xi^2\right)\widetilde{w}_0^{\mathrm{II}} &= \frac{\mathrm{d}\widetilde{p}_0^{\mathrm{II}}}{\mathrm{d}z}, \\ \mu\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \xi^2\right)\widetilde{u}_1^{\mathrm{II}} &= -\xi\widetilde{p}_0^{\mathrm{II}}. \end{aligned}$$

The solutions of this system which vanish at large values of z are

$$\begin{aligned} \widetilde{w}_0^{\mathrm{II}} &= [A(\xi) + \xi z B(\xi)]\,\mathrm{e}^{-\xi z}, \\ \widetilde{u}_1^{\mathrm{II}} &= [A(\xi) - B(\xi) + \xi z B(\xi)]\,\mathrm{e}^{-\xi z}, \\ \widetilde{p}_0^{\mathrm{II}} &= 2\mu\xi B(\xi)\mathrm{e}^{-\xi z}. \end{aligned}$$

The functions $A(\xi)$ and $B(\xi)$ are determined from the appropriate Hankel transforms of the boundary conditions (3.19). This procedure yields

$$A(\xi) = -\mathscr{H}_0[w^{\mathrm{I}}(z=0);\varpi \longmapsto \xi] \tag{3.21}$$

as well as a comparable expression governing $B(\xi)$.

In principle, forming the inverse transforms of (3.21) provides the expressions for $(\boldsymbol{v}^{\mathrm{II}}, p^{\mathrm{II}})$. However, this detailed procedure proves unnecessary, as only the values at $z=0$ are required for the evaluation of F^{II}. Since substitution of $z=0$ commutes with the application of the inverse Hankel transform, the integrand in (3.15) is given by $-2\mu\mathscr{H}_0[\xi A(\xi)|\xi \longmapsto \varpi]$. Substitution of (3.9*a*) evaluated at $z=0$ into (3.22), in conjunction with the identity,

$$\mathscr{H}_n\left[\varpi^n \mathrm{e}^{-\varpi^2/b^2};\varpi \longmapsto \xi\right] = \left(\frac{1}{2}b^2\right)^{n+1} \xi^n \mathrm{e}^{-\xi^2 b^2/4},$$

furnishes the following expression:

$$F^{\mathrm{II}} = \frac{\mu l^3}{16(\pi\alpha t)^{3/2} t} \exp\left(\frac{-Z_-^2}{4\alpha t}\right) \int_{z=0} \mathrm{d}A\, \frac{\varpi Z_-}{\alpha t} \exp\left(\frac{-\varpi^2}{4\alpha t}\right). \tag{3.22}$$

4. Particle motion

When performing the integration in (3.11) and (3.23), with $\mathrm{d}A = 2\pi\varpi\,\mathrm{d}\varpi$, the upper integration limit is set to $\varpi = \infty$. This procedure yields the combined force,

$$F^{\mathrm{I}} + F^{\mathrm{II}} = \frac{\mu l^3}{4\alpha t^2} Z_- \exp\left(\frac{-Z_-^2}{4\alpha t}\right), \tag{4.1}$$

contributed by flows I and II. (The convergence of the integrals supports the validity of replacing $\mathscr{S}$ with $\overline{\mathscr{S}}$.) Flow field III represents a quasi-steady translation of a spherical particle in an incompressible viscous fluid. Accordingly, it contributes the classical Stokes drag force, $\boldsymbol{F}^{\mathrm{III}} = \hat{\boldsymbol{z}}\, F^{\mathrm{III}}$, with $F^{\mathrm{III}} = 6\pi a\mu U$.

The condition of a force-free particle, $F^{\mathrm{I}} + F^{\mathrm{II}} + F^{\mathrm{III}} = 0$, together with the closure relation (2.7), furnish the following ordinary differential equation governing $Z_-(t)$:

$$\frac{\mathrm{d}Z_-}{\mathrm{d}t} = -\frac{l^3}{24\pi a\alpha}\frac{Z_-}{t^2}\exp\left(\frac{-Z_-^2}{4\alpha t}\right). \tag{4.2}$$

Since $\mathrm{d}Z_-/\mathrm{d}t$ is opposite in sign to Z_- (which is initially negative), it is clear that the particle moves in the positive Z-direction. The position $Z_- = 0$ is an equilibrium point, which was to be expected from symmetry arguments: at this position the negative pole P coincides with the centre of the spherically symmetric temperature profile. Note however that P does not necessarily reach the coordinate $Z = 0$, since at $t \to \infty$ any value of Z_- is a possible equilibrium point. Accordingly, $-d \leqslant Z_-(\infty) \leqslant 0$. This implies that $Z_- \ll a$ for all times, *a posteriori* confirming the consistency of approximating $\overline{\mathscr{S}}$ by a moving plane.

Equation (4.2) introduces the time scale $l^3/a\alpha$ at which the transient motion is expected to take place. This, however, is not an exclusive characteristic time: the exponential term introduces yet another scale, d^2/α, which depends upon the initial condition. When analysing equation (4.2) it is convenient to employ dimensionless notation, using the normalized displacement and time variables $\zeta = -Z_-/d$ and $\tau = 24\pi a\alpha t/l^3$. The nonlinear equation (4.2) then takes the form

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} = -\frac{\zeta}{\tau^2}\exp\left(\frac{-\zeta^2}{\gamma\tau}\right), \tag{4.3}$$

in which the parameter γ is proportional to the ratio of the two time scales identified above:

$$\gamma = \frac{1}{6\pi}\frac{l^3}{ad^2}. \tag{4.4}$$

As such, it represents the combined effect of the heat source magnitude Q and the proximity distance d: the larger is γ, the more dominant is the expansion effect in the vicinity of $\overline{\mathscr{S}}$. Equation (4.3) is to be solved subject to the initial condition $\zeta(0) = 1$. Note that the exponential factor becomes $O(1)$ only at $\tau \sim O(\gamma^{-1})$, corresponding to $t \sim O(d^2/\alpha)$.

For short times, $\tau \ll \gamma^{-1}$, the right-hand side of (4.3) is exponentially small, and ζ is close to unity:

$$\zeta \sim 1 + h(\tau), \quad h \ll 1. \tag{4.5}$$

The leading-order correction $h(\zeta)$ is governed by the equation

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} \sim -\frac{\mathrm{e}^{-1/\gamma\tau}}{\tau^2}$$

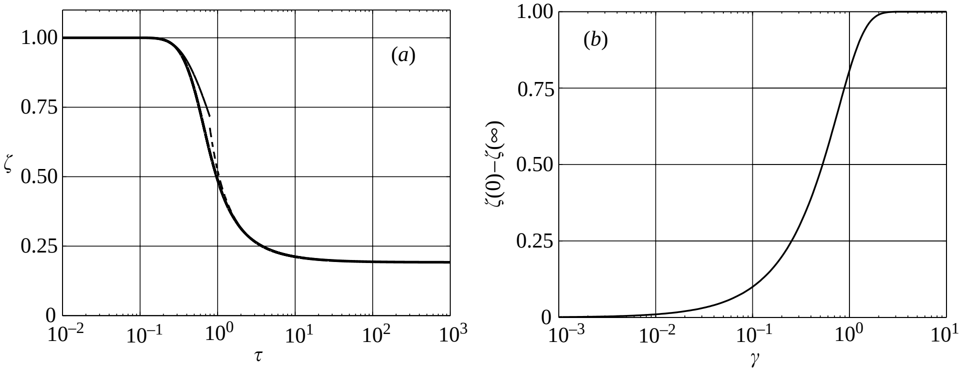

FIGURE 2. (a) Evolution of particle position with time for $\gamma = 1$ (thick solid line); short-time approximation (thin solid line); long-time approximation (dashed line). (b) Total particle displacement as a function of γ.

together with the initial condition $h(0) = 0$. Simple integration yields the following expression governing the short-time evolution of $\zeta(\tau)$:

$$\zeta \sim 1 - \gamma \mathrm{e}^{-1/\gamma\tau}. \tag{4.6}$$

This 'frozen' stage is associated with the initial propagation of the temperature profile, which is exponentially slow.

For large times, $\tau \gg \gamma^{-1}$, the exponential term in (4.3) is approximately equal to unity, hence $\mathrm{d}\zeta/\mathrm{d}\tau \sim -\zeta/\tau^2$. The final approach to $\zeta(\infty)$ is therefore described by

$$\zeta \sim \zeta(\infty)\mathrm{e}^{1/\tau}. \tag{4.7}$$

Because of the singular behaviour of $\zeta(\tau)$ at $\tau = 0$ and $\tau = \infty$, no Taylor series expansion exists about these points. Thus, the unknown constant $\zeta(\infty)$ cannot be determined by any rational matching procedure (Van Dyke 1964), and is only obtainable by numerical integration of (4.3). A typical evolution of $\zeta(\tau)$, obtained via numerical integration of (4.3) for $\gamma = 1$, is depicted in figure 2(a) together with the asymptotic approximations (4.6) and (4.7).

As γ is increased the expansion effects appear earlier, and are therefore of stronger magnitude. For $\gamma \gg 1$, the final stage (4.7) is reached at $\tau \ll 1$, when ζ is still close to unity. Thus, $\zeta(\infty)$ must be exponentially small. Accordingly, as $\gamma \to \infty$ the time evolution of the particle's position approaches an inverted step function, $\zeta(\tau) \to -H(\tau)$, and $\zeta(\infty) = 0$. In the other limit, of small γ, the hydrodynamic localized thrust which acts upon the particle in the neighbourhood of P occurs later and is therefore of weaker intensity. For $\gamma \ll 1$, the final stage (4.7) is reached only at $\tau \gg 1$, when $\zeta \simeq 1$ and $\mathrm{d}\zeta/\mathrm{d}\tau \simeq 0$. Accordingly, $\zeta \equiv 1$ for $\gamma \to 0$. The dependence of the dimensionless net displacement of the particle upon γ is presented in figure 2(b).

5. Concluding remarks

In this paper we have discussed flows generated by time-dependent density fields which are animated by unsteady heating. Specifically, we have focused upon the possibility of propelling small particles in an unbounded liquid. In an attempt to analyse a simple model problem we considered an idealized situation, employing the following simplifying assumptions: (i) weak thermal forcing; (ii) a spherical particle possessing uniform thermal properties, which are identical to those of the fluid;

(iii) large Prandtl number; and (iv) an impulsive and concentrated heating process, occurring in proximity to the particle boundary. This model leads to a nonlinear ordinary differential equation governing the time-evolution of the particle position. The particle tends to move so as to bring its surface into coincidence with the initial heat release point. As expected, this tendency becomes more pronounced as either more heat is released or it is discharged closer to the surface.

It is of interest to examine the limitations posed by the simplifying assumptions. Assumption (i) is supported by the small values of the thermal expansion coefficient in liquids. Assumptions (ii) and (iii) are not strictly valid, but relaxing them would result in a more complicated model, which would only modify the qualitative trends predicted by the present calculation. The most serious limitation to the validity of the present model is assumption (iv), since realistic heating processes possess a characteristic time scale (or frequency) as well as a finite spatial excess. Nevertheless, the present problem captures all the physical elements essential for propulsion by unsteady heating.

The mechanism discussed herein focuses on unsteady dilatation engendered by time-dependent temperature fields, and is independent of gravity. Since free convection is also present in non-isothermal fluid systems, it is of interest to compare the flow magnitude associated with the two different mechanisms, say for a heating process characterized by a frequency ω. The flow magnitude resulting from the unsteady dilatation is set by the mass-balance equation to be $O(a\omega\Delta\rho/\rho_\infty)$, where $\Delta\rho$ is a characteristic density variation (that is, $\Delta\rho/\rho_\infty = O(\beta\Delta T)$, with ΔT being a typical temperature difference associated with the unsteady heating). The buoyancy-driven velocity scale, on the other hand, reflects a dynamic balance between gravity and viscous friction, and is therefore $O(ga^2\Delta\rho/\mu)$, with g being the acceleration due to gravity. The ratio of the former to the latter scales as $\nu\omega/ga$. (Note that this estimate is independent of the heating magnitude.) Thus, at short length and time scales the 'unsteady' contribution dominates over the buoyancy effect. Considering water as the carrying fluid, we find that the two effects become comparable for $\omega = O(10^5 a)$, with ω measured in inverse seconds and a in centimetres. This relation provides a rough estimates of the time scales at which the 'unsteady' effect becomes noticeable.

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